A Survey of Methods for the Two Point Seismic Travel Time Calculation

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Abstract

We have surveyed the literature to evaluate approaches to the seismic two-point travel time problem in which one determines the travel time between two geometrically separated points in three dimensions in a medium with a velocity field that may vary both heterogeneously and anisotropically. Our goal here is to evaluate the various methodologies for implementation in a tool that should be fast, accurate, scalable and extensible to the most general specification of velocity field and non-Cartesian grids.

I. INTRODUCTION

There are a number of approaches of varying complexity to determine the travel time of a seismic wave moving between two points in three dimensions. Two broad classes of methods emerge from the literature. The first is a class of methods based on traditional ray tracing techniques referred to as "shooting and bending" methods. The second set of methods attempts to directly solve the wave equation either in its high frequency approximation as the Eikonal equation or its full representation. The Eikonal equation and the full wave solution determine travel times throughout the region of interest. In the following brief survey we discuss the seminal work and subsequent evolution of these two broad classes of methods focussing attention on scalability and the ability to treat an anisotropic velocity field on a general grid.

Perhaps the earliest example of a two point travel time problem is the well-known Brachistochrone problem first proposed by Johann Bernoulli in 1696. Bernoulli's problem was to find

"...the curve between two points that is covered in the least time by a body that starts at the first point with zero speed and is constrained to move along the curve to the second point, under the action of constant gravity and assuming no friction."

The simplicity with which the problem can be stated belies the elegance of the mathematics that its solution entails. Solutions to this problem were presented by Bernoulli himself as well as such luminaries of the time as Newton, Leibnitz and later Euler. Euler's approach is perhaps the most elegant and laid the foundation for the calculus of variations. The Brachistochrone problem involves movement in an analytic velocity field that is a function only of vertical height, z. The velocity field in problems of interest is almost always non-analytic and more general numerical techniques are required for solutions.

II. RAY TRACING APPROACHES

A. Shooting Methods

Among the first methods that were applied to the two point travel time problem in a general velocity field are the intuitive shooting and bending methods. The shooting method involves 'shooting' rays from point one and adjusting the direction until a ray passes through

point two. Julian [1] published work describing a method for determining ray propagation in inhomogeneous media subsequently used by Engdahl[2] for earthquake analysis. One can imagine two spheres centered on point one. The first is infinitesimally small and the second has a radius which passes through point two. Every ray emerging from point one maps a specific solid angle (θ, ϕ) on the small sphere to a solid angle (θ', ϕ') on the large sphere. The mapping can be many to one to account for multiple paths leading to the same exit point. The converse is not true as there may well be shadow regions that obscure points on the large sphere from receiving any rays from point one. Numerical approaches to the shooting method are fairly straightforward and one can imagine discretizing the velocity field and marching off in a particular direction, adjusting the ray direction based on the local velocity field encountered at each step. Shooting methods suffer from several deficiencies including an inability to consider diffraction paths and paths around shadow zones.

B. Bending Methods

A related approach known as bending methods partially remedies these deficiencies. In the bending approach, one begins with a postulated path between the two points. For example the most simple start might be a straight line. The travel time on the initial path is calculated and perturbations on the path are explored until the minimum time path is found. Advantages of the bending method include generally superior robustness and the ability to explore paths that shooting methods cannot treat. Shooting and bending methods however both suffer from the propensity towards overestimation of travel times as they can settle into local minima. In the space of all possible rays between two points there may be many local minima any one of which may serve as a solution if it is close to the initial guess. Early work on the bending method was contributed by Wesson [3, 4] and Pereyra[5]. The methods focus on solving the two point boundary value problem subject to contraints. The latter developed a code, PASVA3, for 3D raytracing in heterogeneous isotropic media. The approach of Pereyra is to convert the Eikonal equation into coordinates along the ray and to solve a coupled set of three non-linear partial differential equations by reducing them to first degree equations with constraints. Discretization results in a matrix equation.

Julian and Gubbins [6] studied both shooting and bending methods and found the bending method generally superior in efficiency and flexibility. While they judged it to be compu-

tationally intense, it was from the perspective of compute capability circa 1977. Thurber and Ellworth[7] improved the speed of the bending approach by using a one-dimensional approximation through lateral averaging to make a better initial guess. Moser[8] takes a unique approach which maps the problem to the shortest distance through a network. The method places no restrictions on dimension or complexity and is $O(N)$ with N the number of points. Moser's method is constrained to the directions in and between grid points and does not appear to have been developed further by subsequent research groups.

A second approach to bending methods due to Um and Thurber [9] and Prothero[10] minimizes travel time directly through an integral expression. Moser[11] improves on this approach using beta splines along the bending ray and a general conjugate gradient minimization that he claims is more stable than the differentiation technique of Pereyra and others.

III. EIKONAL EQUATION SOLUTIONS

An alternative to traditional ray tracing methods which avoids the deficiency of local minima traps is to solve the high-frequncy representation of the wave equation known as the Eikonal equation. Solution of the Eikonal equation yields a function which is the travel time to every point in the domain of interest. If we consider a large set of two point problems between a group of N_s sources and N_r receivers, then we need only solve $min[N_s, N_r]$ problems. Since it solves for the full domain it is significantly more computationally intense than ray tracing. A number of approaches to the solution of the Eikonal equation have been cited in the literature including wavefront construction methods, direct finite difference, fast marching methods, dynamic programming and others . The principle difference between these approaches lies in how they deal with the general difficulty of multi-valued solutions, cusps and discontinuities and preservation of entropy. The following methods arise from the intuitive realization that determining the travel time in a physical system would be no difficulty at all if we could merely run the experiment. Dropping a pebble in a pond for example sends waves rippling outward which impact surrounding points at specific arrival times. If we could simulate this motion somehow the results would yield the travel times to all points of interest.

A. Wavefront Construction Methods

Wavefront construction methods fit somewhere between ray tracing and finite difference approahes. Instead of tracing individual rays, these methods track the full wavefront from timestep to timestep. Vinje et. al. [12] introduced an approach which essentially constructs the wavefront for two dimensional problems at successive timesteps. The method adds rays to the construction as needed to compensate for geometric dispersion. Vinje[13] extended these methods to three dimensions for smooth, isotropic velocity fields. The extension to three dimensions significantly complicates the method which must now parametrize an expanding surface in three dimensions. In contrast to the finite difference approaches discussed below this method constructs the wavefront in free space on its own grid. The finite difference methods expand outward calculating traveltimes on grid points in successively expanding travel time ranges.

B. Finite Difference Methods

Vidale[14] introduced one of the first attempts at a finite difference approach in 1988. Vidale illustrates his algorithm on a two dimensional grid and shows how, beginning at point zero, travel times to all subsequent surrounding points are generated using interpolation formulas. The approach to preserving causality, i.e. only using points with lesser travel time to extrapolate forward, is somewhat adhoc however and leads to instabilities. The 1990's saw fertile development of these finite difference methods. Vidale extended his original work to three dimensions [15]. Other contributions to finite difference methods were made by Van Trier and Symes[16], Popovici[17, 18] and Schneider[19].

Dynamic programming methods are commonly employed to find minimal distances measured by a given metric on a grid. In Bio-informatics for example the Needleman-Wunsch algorithm provides the minimal edit distance between two strings of genetic material. The method forms a matrix by laying one string horizontally and one string vertically. Values are assigned to each grid point according to an algorithm that takes the minimum of three possible calculations that arise from the three nearest neighbors to the upper left. Variations of these methods can be applied to the two point travel time problem as demonstrated by Schneider[22] who treated an isotropic two dimensional problem. Advantages of Schneider's

method appear to be the ability to treat arbitrary velocity fields in contrast to some of the methods that evolved from Vidale's finite difference treatment. Kumar et. al. [23] extend the work of Schneider to TTI media. They demonstrate their approach in a two dimensional method which calculates travel times directly essentially expanding the wavefront in an anisotropic media using a set of interpolation schemes that are brought to bear.

Building on the work of Vidale, Sethian[24, 25] introduced the fast marching method, an unconditionally stable $O(NLog(N))$ method applicable to any orthogonal coordinate system and arbitrary velocity fields. The problem of calculating first arrival travel times is congruent to that of propagating a wavefront forward normal to its interface with velocity given by the local conditions. Sethian[26] points out the fundamental difficulty with this approach in the formation of cusps and discontinuities that result in instabilities. His solution to this problem is aimed at preserving causality such that points in space are only passed once by the wavefront. Sethian and Popovici[27] apply the fast marching method to geophysical problems. The central innovation in fast marching methods is to solve the Eikonal equation by systematically constructing travel times over all space using an upwind stencil to insure that information travels only one way.

A variation of fast marching methods is the fast sweeping method[28] which can be applied to unstructured grids and general velocity fields and is shown to scale linearly with N. The fast sweeping method solves the Hamilton-Jacobi equation, a generalization of the Eikonal equation. The algorithm proceeds by i)sorting all nodes according to a prescribed metric distance from a few reference points ii) assign large positive values to all vertices except those at the initial front. The boundary notes are assigned exact values and these values are fixed in later iterations iii) use Gauss-Seidel iterations with alternating sweeping orderings according to the distances of nodes to the chosen reference points.

In anisotropic media, the group velocity is not necessarily perpendicular to the wavefront and thus the extension of finite difference methods in such media is not straightforward. The governing equations are more complicated and stability problems arise from the non-colinearity of the phase and group velocity. Qian and Symes[20] are among the first to address anisotropy in finite difference based methods and show that quasi-compressional ("qP") waves are a stable generalized solution to the Eikonal equation and therefore computable. Wang et. al. [21] extend wavefront expansion methods to anisotropic media with the introduction of significant complexity to the update algorithm.

IV. FULL WAVE EQUATION SOLUTIONS

Another approach to this problem and the most computationally intense is the simulation of the full wave equation. In its simplest form applied to an isotropic velocity field this implies an acoustic solution computing pressure as a function of time over the whole physical domain[29]. Finite difference, finite element and pseudo-spectral approaches to this problem have all been demonstrated(see Igel et al[30] and Carcione[31]. Since the earth's sub-surface is more accurately modeled by an anisotropic velocity field, wave equation solutions that incorporate anisotropy were soon required. Generalizations of the isotropic acoustic solution that approximate full anisotropy include pseudo-acoustic formulations that treat VTI and TTI velocity parameterizations. These formulations are typically solved with a finite difference stencil that updates the entire grid each time step. Alkhalifah[32, 33] for example has implemented a finite difference p-wave pseudo-acoustic solution for VTI. Zhang et al[34], Zhou[35, 36], Zhang and Zhang [37] and Fletcher[38] formulate the equivalent for TTI. Anisotropic pseudo-acoustic approaches are prone to instabilities but methods have emerged that result in stable solutions. The next logical step in the state of the art is likely to be the full numerical simulation of the full elastic wave equation using a general elastic tensor with parameters that are a function of three dimensional space. The computational resources required for its solution have thus far been out of reach however with new technologies rapidly developing such as GPU computing it seems certain that numerical solutions to the full elastic wave equation will begin to appear in the literature and in production codes in the next few years. Simulation of the full wave equation, like the solution of the high frequency approximate Eikonal, provides a travel time solution throughout the whole domain of interest for each source considered.

V. DISCUSSION AND ANALYSIS

We have investigated the literature on a wide variety of methods and approaches all formulated to solve the two point travel time problem. These include ray tracing approaches such as shooting and bending, variations on ray tracing including wavefront construction, various finite difference approaches including fast marching and fast sweeping methods, dynamic programming and direct finite difference and finally we've included full wave equation

solvers focussing on finite difference methods applied to isotropic, VTI and TTI anisotropy. The goal of this survey is to identify a method that is accurate, fast, scalable and able to treat anisotropic velocity fields and non-Cartesian grids.

We believe that the ray based methods, shooting and bending should not be considered as long term solutions because they will not give deterministic answers to problems. Their results are biased by the initial conditions and are susceptible to local minima. This leaves the full range of methods that solve the wave equation or its high frequency approximate, the Eikonal equation. In the following sub sections we discuss the advantages and disadvantages of Eikonal vs full wave equation solutions across six metrics: performance, scalability, accuracy, velocity anisotropy and non-Cartesian Grids.

A. Performance

The performance metric here concerns the "per-problem" scalability with the number of grid points, N. Considering the Eikonal equation first the methods which appear to scale most favorably are those recently introduced by Qian[28] as the fast sweeping methods. Qian presents them as $O(N)$ methods where N is the number of points in the grid. Since the Eikonal equation gives the solution for many receivers at once for the same source the scaling of a full set of travel time problems over pairs made from a group of N_s sources and N_r receivers is $min[N_s, N_r]$. Total run time would then scale as $O(Nmin[N_s, N_r])$. The full wave equation solver whether isotropic, VTI or TTI will scale like $O(N^{4/3}min[N_s, N_r])$. The extra factor of $N^{1/3}$ accounts for the number of time steps required for the wave to move from source to receiver. Its clear that full wave equation methods are more computationally intense. They yield more information than we actually require for the problem. In addition to first arrival travel time they also provide amplitude and phase information throughout the three dimensional domain for each time step. The extra work done may be moderated somewhat by the fact that full wave solutions are more easily parallelized than Eikonal solvers as discussed in more detail below.

B. Scalability

There is a trivial scalability that results from the fact that we intend to solve not one but many independent problems. There is a different independent problem for each source and each receiver. In practice we have the flexibility to do either N_s or N_r independent problems. We would certainly choose the minimum of these options. In a typical case we may have $N_r \sim 10^2$ and $N_s \sim 10^4$ and we would formulate 10^2 different problems, one for each receiver. This trivial scalability apples to the Eikonal and full wave solution equally.

There is another level of scalability at the "per-problem" level. A-priori one would assume that Eikonal solvers, which incorporate an inherent causality and thus dependency condition, would be more difficult to parallelize in practice. Finite difference full wave approaches by contrast are highly amenable to parallel scalability. Implementation on GPU's for example is by now quite common and almost required to be competitive on performance. However recent advances in Eikonal solvers such as the fast sweeping methods of Qian[28] appear to provide more opportunity to exploit parallelism. Fast sweeping methods update the entire grid multiple times in sweeps which "iron-out" causality. To summarize, both Eikonal and full wave solutions will benefit from task parallelism. At the "per-problem" level however implementations of highly parallel full wave solutions are quite common while more work would need to be done to parallelize Eikonal solvers. In particular the path forward for an Eikonal solver on GPU is not completely clear at this time.

C. Accuracy

Eikonal solvers make a high-frequency approximation to the wave equation and will therefore lead to results that are less accurate to some degree than full wave equation solutions. It should be noted that Eikonal solvers only provide the first arrival travel time while full wave equation treatments would yield all subsequent arrival waves as well.

D. Velocity Anisotropy

The additional complications to the respective formulations needed to treat velocity anisotropy appear roughly equivalent i.e. equally difficult. There are numerous examples in the literature which treat anisotropic velocity fields e.g. VTI and TTI for both.

E. Non-Cartesian Grids

Non-Cartesian grids, e.g. unstructured tetrahedral, can often be interpolated to Cartesian grids. The efficacy of this interpolation is lost however if gridding dimension varies significantly across the problem domain. In the latter case the structured Cartesian grid would need to take on the size of the smallest fine grid dimension to maintain the correct level of resolution. Both methods can be implemented in non-Cartesian grids. Eikonal solvers such as the fast sweeping approach have been formulated to adapt to non-Cartesian grids and preserve causality in the propagation of the wavefront[28]. Finite difference full wave equation solvers are in principle able to handle unstructured grids however they are mainly formulated to orthogonal grids because the application of the finite difference stencil is vastly simplified in those cases. Generalizing to an unstructured grid is possible however it increases the complexity of the formulation and likely reduces the amenability to parallel implementation particularly on GPU. To first order we would predict that it would be slightly easier to apply Eikonal solvers to non-Cartesian grids than full wave solvers.

The choice of methodology to develop depends on the relative weight assigned to the principle criteria above. In the broadest sense if speed is the ultimate issue then at this time the Eikonal solver approach should probably be pursued with the caveat that efficient GPU implementations can be derived. Its scaling behavior is superior and it provides only the information we require. If accuracy and information beyond first time arrival is required then the full wave equation solution is recommended in its TTI formulation. Looking forward hardware capability will be able to support simulation of the full elastic wave equation in the near future and investment in a full wave equation solver now may prove prescient. Numerical implementation of either approach however is non-trivial in all but the simplest isotropic cases.

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